Profit maximization in the short run with Cobb-Douglas

Suppose that in an agricultural establishment the land area available for work is fixed, but production can be increased by increasing the amount of labor applied to it; the problem is that each additional unit of labor is less productive than the previous one. Assuming that this situation can be represented by the production function: $f(\bar{T}, L) = \bar{T}L^{\alpha}$.

1. Find the values of the parameter α consistent with the conditions faced by the agricultural establishment.
2. Find the optimal level of labor that the establishment must choose if the grain price is p, and the wage is given by w. How does L depend on the selling price and the wage?
3. Does the value of the land influence the choice in the previous point?

Solutions

1. For the marginal productivity to be decreasing, it is necessary that $0 < \alpha < 1$ since in this way the derivative:

$$f'_L = \alpha \bar{T} L^{\alpha-1} > 0$$

But also as L increases, the derivative takes a progressively smaller, albeit positive, value, implying diminishing marginal returns regarding labor. This can also be seen with the negative second derivative:

$$f''_{LL} = \alpha(\alpha - 1)\bar{T} L^{\alpha-2} < 0$$

2. We state the profit function:

$$B = p\bar{T}L^{\alpha} - (Lw)$$

$$B'_L = \alpha p\bar{T}L^{\alpha-1} - w = 0$$

$$L = \left(\frac{w}{\alpha p\bar{T}}\right)^{\frac{1}{\alpha-1}} = \left(\frac{\alpha p\bar{T}}{w}\right)^{\frac{1}{1-\alpha}}$$

To see how a price or wage increase affects, we derive:

$$L'_w = \frac{1}{1-\alpha}\left(\frac{\alpha p\bar{T}}{w}\right)^{\frac{1}{1-\alpha}-1}(-1)\frac{\alpha p\bar{T}}{w^{-2}} < 0$$

An increase in wage leads to a decrease in the use of labor. On the other hand:

$$L_p = \frac{1}{1-\alpha}\left(\frac{\alpha p\bar{T}}{w}\right)^{\frac{1}{1-\alpha}-1}\frac{p\bar{T}}{w} > 0$$

An increase in the price of the product being sold causes more labor to be demanded.

3. The value of the land does not influence since it is not present in the demand